On Two-sided Matching in Infinite Markets

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Abstract. Matching is a branch of economic theory that has seen real-life applications in the assignment of doctors to medical residencies, students to schools, and cadets to branches of military services. Although standard matching models are finite, economic theorists often lean on infinite market models as approximations of large market behaviors. While matching in finite markets has been studied extensively, the study of infinite matching models is relatively new. In this paper, we lift a number of classic results for one-to-one matching markets, such as group strategy proofness, comparative statics, and respect for unambiguous improvements, to infinite markets via the compactness theorem of propositional logic. In addition, we show that two versions of the lattice structure of finite markets carry over to infinite markets. At the same time, we prove that other results, such as weak Pareto optimality and strong stability property, do not hold in infinite markets. These results give us a clearer sense about which matching results are the most canonical.

Keywords: Market Design, Matching, Infinite Markets

Contents

1 Introduction

Given a set of men and women and their preferences in the opposite gender,[1] how do we form pairs such that no agent has the incentive to rematch? This question was originally proposed and answered by Gale and Shapley [1] in 1962; since then, the solutions to the variants of the question have been applied to medical residency matching [2], school choice [3], and the assignment of cadets to military branches [4]. Gale and Shapley sought a one-to-one *stable matching*, i.e., a set of pairings in which (1) all men and women find their partners acceptable and (2) no man/woman pair prefers each other to their assigned partners. By a process called *deferred acceptance algorithm*, Gale and Shapley showed that a stable matching exists for any finite set of men and women and their preferences. The algorithm can be described as follows:

1. Each man proposes to his most preferred woman. Each woman selects her most preferred man out of those that proposed to her and holds him; the rest are rejected.

2. Each man that was rejected moves down his list of preferences and proposes to his next preferred woman. Each woman selects her most preferred man among those that proposed to her, including the man that she held from the previous iteration.

3. Step 2 repeats until no men are available to propose.

Gale and Shapley [1] showed that the matching obtained at the termination of the algorithm is stable. For example, consider the economy displayed in Figure 1. Following the steps of the algorithm, we have that (m_1, w_1) and (m_3, w_2) be matched. Casework shows that the matching is stable. In addition, Gale and Shapley showed that each man weakly prefers his match under the deferred acceptance algorithm compared to any other woman that he could have been matched to in another stable matching. Additional properties of

[1]The set up of the model is outdated in today's standards. However, the marriage model has traditionally been the stand-in problem for matchings in which every agent is matched across sides.

$$
\begin{array}{ll}
m_1 : \underline{w_1} \succ w_2 \succ \emptyset & w_1 : \underline{m_1} \succ m_3 \succ m_2 \succ \emptyset \\
m_2 : w_1 \succ w_2 \succ \emptyset & w_2 : m_1 \succ \underline{m_3} \succ \emptyset \\
m_3 : \underline{w_2} \succ \emptyset &
\end{array}
$$

Figure 1: Example of a marriage market with three men, m_1, m_2, and m_3 and two women, w_1 and w_2. Preferences are as denoted where $\emptyset$ represents the option to remain unmatched. Underlined partners indicate matches obtained at the termination of the deferred acceptance algorithm [1].

the man-optimal stable matching are known: first, that the man-optimal stable matching is group strategyproof [5]—no set of agents can be strictly better off by reporting false preferences. Second, the man-optimal stable matching satisfies natural entry comparative statics (see, e.g., [6, 7])—adding new women to the market leaves every man weakly better off. Lastly, the man-optimal stable matching respects unambiguous improvement [8]—when a man's ranking in all the women's preference list improves, then his outcome improves as well.

A key to Gale and Shapley's algorithm [1] is the finiteness of the market, which ensures that the algorithm terminates. In the real world, no matter how large a set of men and women is, it is at most finite. Therefore, we don't lose generality by assuming that our set of agents is finite. However, finite models may fail to fully capture the story of large markets. In particular, the dependence on the finiteness may make us prone to small frictions, perturbations, and arbitrary input data [9]. Moreover, infinite models are understood to be better representations of large finite markets than large finite models, as they allow us to study limit behaviors and to approximate large market behaviors.

The pioneering work in infinite matching models has been completed by Fleiner [10], Azevedo and Leshno [11], Zanardo [12], Jagadeesan [13], and Gonczarowski et al. [9]. Notably, Fleiner gave the first existence proof of stable matchings in infinite markets. Azevedo and Leshno [11] introduced an analog of infinite matching models with a continuum of agents. Zanardo and Jagadeesan, respectively, introduced infinite variants of the deferred acceptance algorithm in countably infinite and locally finite markets. Gonczarowski et al., via logical compactness, proved the existence of man-optimal stable matchings in infinite markets. In addition, they showed that the man-optimal stable matching mechanism in infinite markets is individual strategyproof.

This paper is structured as follows. In Section 2, we introduce the marriage model and the logical framework for analyzing the marriage model. In Section 3, we show via the compactness theorem of propositional logic that the man-optimal stable matching mechanism in infinite markets, like in finite markets, is group strategyproof, satisfies natural entry comparative statics, respects unambiguous improvement, and respects preference extensions. In addition, we show that infinite markets carry over the lattice structures of finite markets. In Section 4, we prove the failure of weak Pareto optimality and strong stability property in infinite markets. In Section 5, we discuss our results. Section 6 is acknowledgments. Proofs omitted from the main text are presented in Section A of the appendix; further results are presented in Sections B through E.

2 Nuts and Bolts

2.1 Marriage Model

In this subsection, we introduce the infinite marriage model, generalizing the model proposed by Gale and Shapley [1]. Suppose that we are given an economy I consisting of agents: (potentially infinite) sets of men M and women W. We say that I is finite if the number of agents is finite and that it is infinite otherwise. We refer to P as the agents' collective set of preferences and use $\succ_i$ to denote the relative preference ordering of $i \in I$ where $\emptyset$ represents the option to remain unmatched. While $|M|$ and $|W|$ can in principle be uncountably infinite, we assume that each agent prefers being matched to countably many agents over $\emptyset$. A matching $\mu : I \to I \cup \{\emptyset\}$ is defined as follows: for each $m \in M$ and each $w \in W$,

1. $\mu(m) \in M \cup \{\emptyset\}$, $\mu(w) \in W \cup \{\emptyset\}$, and

2. $\mu(m) = w$ if and only if $\mu(w) = m$.

In particular, a matching is *stable* if it is

1. *individually rational*, i.e., for any agent $i \in I$, $\mu(i) \succeq_i \emptyset$ and

2. *unblocked*, i.e., there exists no man $m \in M$ and woman $w \in W$ such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

An agent is an *achievable partner* of an agent if there is a stable matching in which the two are matched. In both finite [1] and infinite [9] markets, there always exists a stable matching in which every man is matched to his most preferred achievable partner. Such a matching is denoted as μ_M and is called the *man-optimal stable matching.* μ_W is defined in a similar fashion.

When comparing two stable matchings μ and μ' under the same set of M, W, and P, we say that $\mu \succ_M \mu'$ (respectively, $\succeq_M$) if for every man $m \in M$, $\mu(m) \succ_m \mu'(m)$ (respectively, $\succeq_m$). $\mu \succ_W \mu'$ and $\mu \succeq_W \mu'$ are defined similarly.

2.2 Logical Frame for Analyzing the Marriage Model

In this subsection, we establish the preliminaries on the logical frame for analyzing the marriage model, which will transform the conditions of matching into logical statements. A *formula* is a boolean statement that can be defined inductively from an atomic boolean statement, ϕ. The inductive construction of a formula is as follows: if ϕ and ψ are formulae, then so are $\neg\phi$, $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \to \psi$, and $\phi \leftrightarrow \psi$.[2] A well-formed formula can be arbitrarily long; however, it has to be finite in length. The compactness theorem of propositional

[2]Here, we use the standard notations: $\neg$ for not, $\vee$ for or, $\wedge$ for and, $\to$ for implies, $\leftrightarrow$ for if and only if.

logic [14] states that an infinite set of individually finite logical formulae can be satisfied concurrently if and only if any finite subset of the infinite set of formulae can be.

In the rest of this section, we establish the set of formulae Φ as illustrated in Gonczarowski et al. [9] that characterizes stable matchings. We first introduce $\mathsf{matched}[m, w]$, which is TRUE when m and w are matched and FALSE otherwise. Then, for every pair of men and women (m, w) and a woman $w'(\neq w)$, we introduce the following formula:

$$\mathsf{matched}[m, w] \rightarrow \neg\mathsf{matched}[m, w'] \in \Phi,$$

which ensures that every man is matched to at most one woman. Similarly, for every pair of men and women (m, w) and a man $m'(\neq m)$, we introduce the following formula:

$$\mathsf{matched}[m, w] \rightarrow \neg\mathsf{matched}[m', w] \in \Phi,$$

which ensures that every woman is matched to at most one man. In addition, we add the following formula for any pair (m, w) such that at least one finds the other incompatible:

$$\neg\mathsf{matched}[m, w] \in \Phi,$$

which guarantees that no incompatible man and woman are matched. Lastly, the following formula guarantees that the matching is unblocked. For every pair of man and woman (m, w) in which both of them find each other acceptable, let $w_1, \ldots, w_\ell$ denote the finite set of women that m prefers over w, and let $m_1, \ldots m_k$ denote the finite set of men that w prefers over m. We include the following formula:

$$\neg\mathsf{matched}[\mathsf{m}, \mathsf{w}] \rightarrow \mathsf{matched}[m, w_1] \vee \mathsf{matched}[m, w_2] \vee \ldots \vee \mathsf{matched}[m, w_\ell]$$
$$\vee \mathsf{matched}[m_1, w] \vee \mathsf{matched}[m_2, w] \vee \ldots \vee \mathsf{matched}[m_k, w] \in \Phi,$$

which guarantees that if m and w are not matched, then at least one of m or w is matched to an agent that they find more preferable to w or m, respectively. This, in turn, means that no unmatched (m, w) prefer each other to their respective partners. The length of the formula is finite as each agent only finds countably infinite or finitely many agents acceptable.

By construction, any matching satisfying all the formulae in Φ is a matching that is individually rational and block-free, and thus stable. Using Φ and additional formulae, we can show the existence of certain stable matchings via appeal to the results of finite markets.

3 Generalizations to Infinite Markets

In this section, we use logical compactness to lift a number of classical results of matching from finite to infinite markets. Notably, we will show that the man-optimal stable matching mechanism is group strategyproof (Theorem 3.3), satisfies natural entry comparative statics (Theorem 3.5), respects unambiguous improvement (Theorem 3.7), and respects preference extensions (Theorem 3.10). Before proceeding to prove these results, we prove the following lemma that lets us project the stable matchings of infinite markets to finite markets.

Lemma 3.1. *Let μ be a stable matching in I, and let M' and W' be finite subsets of M and W. Take the finite economy I' consisting of M', W', $\mu(M')$, and $\mu(W')$. Then, μ is a stable matching in I'.*

As μ is individually rational and block-free in I, we check that μ is stable in I' as well.

3.1 Group Strategyproofness

We extend the *group strategyproofness* of stable matching mechanisms of finite markets to infinite markets. Group strategyproofness states that no set of agents can be strictly better off by reporting false preferences. Group strategyproofness is important for the design of stable matching mechanisms in practice, as it reduces the agents' incentives to gain from the system in ways that distort the outcome. In finite markets, the statement of group strategyproofness [5, 7] is formalized as follows:

Theorem 3.2 (G. Demange et al. [15], Hatfield-Kojima [7])**.** *In a finite, one-to-one matching market, no agents $\acute{I}$ can manipulate their choices to produce a stable matching that is strictly preferable for all agents in $\acute{I}$ compared to some stable matching under true preferences.*

A special case of Theorem 3.2 is *individual strategyproofnes*, which states that a market is strategyproof against an individual manipulating his or her preferences. Individual strategyproofness was proven to hold in infinite markets by Gonczarowski et al. [9]. Here, we show that group strategyproofness holds in infinite markets.

Theorem 3.3. *In a (potentially infinite), one-to-one matching market, no finite set of agents* $\acute{I}$ *can manipulate their choices to produce a stable matching that is strictly preferable for all agents in* $\acute{I}$ *compared to some stable matching under their true preferences.*

We generalize the proof of Gonczarowski [9] using logical compactness. The key divergence is that we use a different formula to characterize improvements.

Remark. The conclusion of Theorem 3.3 does not hold when $|\acute{I}|$ is infinite; this is a direct consequence of our Theorem 4.4, which shows the failure of the weak Pareto optimality in infinite markets.

3.2 Entry Comparative Statics

The next result in finite markets to be generalized is the entry comparative statics, which states that when new women enter the economy and each man updates their preferences accordingly, the outcome for every man under the man-optimal stable matching weakly improves. This result helps us understand how match outcomes change as market participation changes over time. The formalized statement in the finite market [6, 16] is as follows:

Theorem 3.4 (Kelso-Crawford [6], Gale-Sotomayor [16])**.** *In a finite, one-to-one market, if a new set of women* $\overline{W}$ *enters, then the outcome of every man in the man-optimal stable mechanism weakly improves.*

An isomorphic model to the market before the entrance of $\overline{W}$ assumes that those women are present, but that they find no man acceptable [17]. In this model, the women in $\overline{W}$ are not matched as if they are absent, because they find no one acceptable. Thus, we may assume that the men in this model rank all women, including the ones in $\overline{W}$. When the women of $\overline{W}$ moves $\emptyset$ down their preferences and report their true preferences, the model is isomorphic to when $\overline{W}$ enters the market. Notice that the men's preferences do not change. With this in mind, we generalize Theorem 3.4 to infinite markets.

Theorem 3.5. *In a (potentially infinite), one-to-one market, if a new (potentially infinite) set of women $\overline{W}$ enters, then the outcome of every man in the man-optimal stable mechanism weakly improves.*

Proof. We say that $\widehat{M}$ is the set of men that gets matched to some woman in the man-optimal matching prior to the update, which we call μ_M. We want to show that when $\overline{W}$ enters, then each man in $\widehat{m} \in \widehat{M}$ gets matched to a woman at least as preferable as $\mu_M(\widehat{m})$.

We introduce a new set of formulae Φ' that contain all the formulae in Φ, and in addition, the following formulae for each $\widehat{m} \in \widehat{M}$:

$$\vee_{w \succeq_{\widehat{m}} \mu_M(\widehat{m})} \mathsf{matched}[\widehat{m}, w], \tag{1}$$

which ensures that each $\widehat{m} \in \widehat{M}$ is matched to a woman at least as preferable as $\mu_M(\widehat{m})$. The length of Eq. (1) is finite as $\widehat{m}$ can each only prefer finitely many woman over $\mu_M(\widehat{m})$.

Now, take a finite set of formulae from Φ' and the resulting finite economy I' consisting of M', W', $\mu_M(M')$, $\mu_M(W')$ and $\overline{W}$, where M' and W' are the finite sets of men and women mentioned in our finite set of formulae. Now, we will show that the man-optimal stable matching in I' after $\overline{W}$ enters, which we denote as μ'_M, satisfies our finite set of formulae. Since we already know that such a matching is stable, we only need to show that the matching satisfies Eq. (1) for all men in $M' \cap \widehat{M}$. Now, we know from Lemma 3.1 that μ_M is a stable matching in I' before the entrance of $\overline{W}$. Therefore, the man-optimal stable matching in I' before the entrance of $\overline{W}$ should match each $m \in M' \cap \widehat{M}$ to a woman at least as preferable as $\mu_M(m)$. Note that the entrance of $\overline{W}$ creates the same effect in I' as it does in I: the preference list of the women in $\overline{W}$ changes from $\emptyset$ to their true preferences. Now, because

I' is a finite economy, by Theorem 3.4, μ'_M is weakly preferable to the man-optimal stable matching before the entrance of $\overline{W}$ for all the men in I'. We have already established that before $\overline{W}$'s entrance, the man-optimal stable matching matches every man in $m \in M' \cap \widehat{M}$ to a woman at least as preferable as $\mu_M(m)$. Therefore, under μ'_M, each $m \in M' \cap \widehat{M}$ should be matched with a woman at least as preferable as $\mu_M(m)$ as well. From here, we conclude that such a matching satisfies Eq. (1)—and thus, fully satisfies our finite set of formulae. Logical compactness states that if there is a model that satisfies any finite set of formulae, then there is a model that satisfies every formula in our infinite set; hence, we are done. $\square$

3.3 Unambiguous Improvement

We say that the rankings of a man *unambiguously improve* when in each woman's preferences, his rankings weakly improve while leaving the relative rankings of the others static. The next result that we extend from finite markets is that the man-optimal stable matching mechanism *respects unambiguous improvement*: when the rankings of a particular man unambiguously improve, then his outcome under the man-optimal stable matching weakly improves. In practice, the respect of unambiguous improvement means that agents have the incentive to improve their rankings. The statement in finite markets [8] is as follows.

Theorem 3.6 (Balinski-Sonmez, [8])**.** *In a finite, one-to-one market, the man-optimal stable mechanism respects unambiguous improvement. That is, if $\overline{m}$ unambiguously improves in the preferences of W, then his matching under μ_M weakly improves.*

We generalize Theorem 3.6 to infinite markets via logical compactness.

Theorem 3.7. *In a (potentially infinite), one-to-one market, the man-optimal stable mechanism respects unambiguous improvement. That is, if $\overline{m}$ unambiguously improves in the preferences of W, then his matching under μ_M weakly improves.*

Like the proofs of Theorem 3.3 and Theorem 3.5, we project the man-optimal stable matchings before and after the unambiguous improvement of $\overline{m}$ into a finite market and conclude by invoking the result of the finite market.

3.4 Impact of Preference Extensions

The next result to be lifted from finite to infinite markets explains how outcomes change if men become less selective. Before stating and proving the statement, we establish Lemma 3.8, originally established by Knuth in finite markets [18]. We note here that his argument directly generalizes to infinite markets as it relies solely on the stability of μ and μ'.

Lemma 3.8. *In a (potentially infinite), one-to-one market, if μ and μ' are two stable matchings, then we have that $\mu \succeq_M \mu'$ if and only if $\mu' \succeq_W \mu$.*

The statement on the impacts of preference extension in finite markets is [16] as follows:

Theorem 3.9 (Gale and Sotomayor [16])**.** *In a finite, one-to-one market, suppose that the men extend their list of preferences to $\tilde{P}$ such that they each add (a potentially empty set of) additional women to the end of their list of acceptable women. Let $\tilde{\mu}_M$ and $\tilde{\mu}_{\tilde{W}}$ denote the man and woman-optimal stable matchings under $\tilde{P}$. Then, we have that*

$$\mu_M \succeq_M \tilde{\mu}_M \text{ (consequently, } \tilde{\mu}_M \succeq_W \mu_M \text{ by Lemma 3.8) and}$$

$$\tilde{\mu}_W \succeq_W \mu_W \text{ (consequently, } \tilde{\mu}_W \succeq_M \tilde{\mu}_W \text{ by Lemma 3.8) under } P.$$

To generalize Theorem 3.9 to infinite markets, we use logical compactness.

Theorem 3.10. *In a (potentially infinite), one-to-one market, suppose that the men extend their list of preferences to $\tilde{P}$ such that they each add (a potentially empty set of) additional women to their list of acceptable women such that the newly added women are less preferable than the women that they initially found acceptable. Let $\tilde{\mu}_M$ and $\tilde{\mu}_{\tilde{W}}$ denote the man and* *woman-optimal stable matchings under $\tilde{P}$. Then, we have that*

$$\mu_M \succeq_M \tilde{\mu}_M \text{ (consequently, } \tilde{\mu}_M \succeq_W \tilde{\mu}_M \text{ by Lemma 3.8) and}$$

$$\tilde{\mu}_W \succeq_W \mu_W \text{ (consequently, } \mu_W \succeq_M \tilde{\mu}_W \text{ by Lemma 3.8) under } P.$$

3.5 Lattice Theorem

The last result to be extended is the lattice theorem. For two stable matchings μ and μ', let $\check{\lambda} = \mu \vee \mu'$ such that $\check{\lambda}(m) = \mu(m)$ if $\mu(m) \succ_m \mu'(m)$, and $\check{\lambda}(m) = \mu'(m)$ otherwise.[3] Similarly, define $\hat{\lambda} = \mu \wedge \mu'$ such that each man points to his less preferable partner and each woman points to her more preferable partner. In finite markets, for stable matchings μ and μ', both $\mu \vee \mu'$ and $\mu \wedge \mu'$ are stable matchings. Put it differently, the stable matchings form a lattice with their join as $\wedge$ and meet as $\vee$. Consequently, by repeatedly executing $\vee$ and $\wedge$, the maximal and minimal elements of the lattice can be found. Conway was the first to formalize this statement for finite marriage models [18].

Theorem 3.11 (Conway [18])**.** *In a finite, one-to-one market, for stable matchings μ and μ', $\mu \vee \mu'$ and $\mu \wedge \mu'$ are stable matchings. Consequently, the stable matchings form a lattice with μ_M and μ_W as the maximal and minimal elements.*

Adachi [19], among others, interpreted Conway's lattice through the lens of Tarski's Fixed Point Theorem [20]. While Conway's original proof does not generalize to infinite markets, Adachi's setup generalizes in our context.

Theorem 3.12. *In a (potentially infinite), one-to-one market, for stable matchings μ and μ', $\mu \vee \mu'$ and $\mu \wedge \mu'$ are stable matchings. Consequently, the stable matchings form a lattice with μ_M and μ_W as the maximal and minimal elements.*

[3] $\vee$ and $\wedge$ are redefined in this subsection as described. They do not follow the definitions defined in the context of logic in Section 2.

In finite markets, Conway's proof of the lattice theorem [18] is a direct consequence of a decomposition lemma due to Gale and Sotomayor [16]. However, the exact statement of the decomposition lemma does not hold in infinite markets. But proceeding backward from our lattice theorem, we prove a relaxed version of the decomposition lemma.

Lemma 3.13. *In an infinite, one-to-one market, let μ and μ' be two stable matchings. Let $M(\mu)$ denote the set of men that prefer their partners under μ than μ'. Similarly, define $M(\mu')$, $W(\mu)$, and $W(\mu')$. Then, both μ and μ' map $M(\mu') \cup \{\emptyset\}$ onto $W(\mu) \cup \{\emptyset\}$. Similarly, they each map $M(\mu) \cup \{\emptyset\}$ to $W(\mu') \cup \{\emptyset\}$.*

Hatfield and Kominers [21] presented an alternative lattice construction. In Appendix D, we introduce the Hatfield-Kominers' operator and show that their setup generalizes to infinite markets. Although our results so far relied on the countability of the preference list of each agent, we remove this constraint under the infinite Hatfield-Kominers Lattice.

4 Properties that *Do Not* extend

While the previously mentioned results on finite markets extend to infinite markets, others do not. In this section, we will characterize such results via counterexamples.

4.1 Lone Wolf Theorem

The question of whether different stable matching mechanisms yield different sets of men matched to some woman is crucial when matching doctors to hospitals. As rural hospitals are less preferable for many doctors, they often faced a shortage of doctors. To assess whether the matching mechanism was to blame, the lone wolf theorem was studied.

Theorem 4.1 (McVitie-Wilson [22])**.** *In a finite, one-to-one market, the set of men matched to some woman is invariant across all stable matchings.*

Jagadeesan [13] showed that the lone wolf theorem does not hold in infinite markets.

Theorem 4.2 (Jagadeesan, [13])**.** *In an infinite, one-to-one market, the set of men matched to some woman may not be invariant across stable matchings.*

In Jagadeesan's construction, only finitely many men were matched in one stable matching and not in the other. Here, we show that the changeover can in fact be infinite.[4]

Example 4.1. Consider the following infinite market in which the men and women are indexed with positive integers. Suppose that for all $k \in \mathbb{Z}^+$, we have that:

$$m_{2k} : w_{2k} \succ \emptyset,\ m_{2k-1} : w_{2k-1} \succ w_{k-1} \succ \emptyset,\ \text{and}$$

$$w_k : m_{2k+1} \succ m_k \succ \emptyset.$$

In this economy, under μ_M, each man is matched with their top choices, i.e., $\mu_M(m_k) = w_k$ for all $k \in \mathbb{Z}^+$. μ_M is individually rational and stable as each man receives his top choice, which ensures no unmatched men and women mutually prefer each other to their assigned partners. Similarly, under, μ_W each woman receives her top choice, i.e., $\mu_W(w_k) = m_{2k+1}$ for all $k \in \mathbb{Z}^+$. Observe that under μ_M, all agents are matched. However, under μ_W, only men of odd indices are matched. Thus, the changeover is infinite.

Remark. In the finite market, the lone wolf theorem follows from the decomposition lemma [16], but in our decomposition lemma, the addition of $\{\emptyset\}$ fails the lone wolf theorem.

4.2 Weak Pareto Optimality

Another classical result for stable matchings in finite markets is the weak Pareto optimality principle [2], which states that no individually rational matching (not necessarily stable) matches every man to a woman strictly more preferred to their matches under μ_M.

[4] While Jagadeesan's example and Example 4.1 discuss markets with countably infinite agents, we present a related counterexample with uncountably many agents in the appendix.

$$
\begin{array}{ll}
P(m_1) = \underline{w_3}, \boldsymbol{w_4} & P(w_1) = \emptyset \\
P(m_2) = w_3, \underline{w_4}, \boldsymbol{w_5} & P(w_2) = \emptyset \\
P(m_3) = \underline{w_6}, \boldsymbol{w_3} & P(w_3) = \boldsymbol{m_3}, m_2, \underline{m_1} \\
P(m_4) = w_6, \underline{w_7}, \boldsymbol{w_8} & P(w_4) = \boldsymbol{m_1}, \underline{m_2} \\
P(m_5) = w_6, w_7, \underline{w_8}, \boldsymbol{w_9} & P(w_5) = \boldsymbol{m_2}, m_3 \\
P(m_6) = \underline{w_{10}}, \boldsymbol{w_6} & P(w_6) = m_7, \boldsymbol{m_6}, m_5, m_4, \underline{m_3} \\
P(m_7) = \underline{w_{11}}, \boldsymbol{w_7} & P(w_7) = \boldsymbol{m_7}, m_6, m_5, \underline{m_4}, m_3 \\
P(m_8) = w_{10}, w_{11}, \underline{w_{12}}, \boldsymbol{w_{13}} & P(w_8) = \boldsymbol{m_4}, \underline{m_5} \\
P(m_9) = w_{10}, w_{11}, w_{12}, \underline{w_{13}}, \boldsymbol{w_{14}} & P(w_9) = \boldsymbol{m_5}, m_6 \\
P(m_{10}) = \underline{w_{15}}, \boldsymbol{w_{10}} & P(w_{10}) = m_{12}, m_{11}, \boldsymbol{m_{10}}, m_9, m_8, m_7, \underline{m_6} \\
P(m_{11}) = \underline{w_{16}}, \boldsymbol{w_{11}} & P(w_{11}) = m_{12}, \boldsymbol{m_{11}}, m_{10}, m_9, m_8, \underline{m_7}, m_6 \\
P(m_{12}) = \underline{w_{17}}, \boldsymbol{w_{12}} & P(w_{12}) = \boldsymbol{m_{12}}, m_{11}, m_{10}, m_9, \underline{m_8}, m_7, m_6 \\
P(m_{13}) = w_{15}, w_{16}, w_{17}, \underline{w_{18}}, \boldsymbol{w_{19}} & P(w_{13}) = \boldsymbol{m_8}, \underline{m_9} \\
P(m_{14}) = w_{15}, w_{16}, w_{17}, w_{18}, \underline{w_{19}}, \boldsymbol{w_{20}} & P(w_{14}) = \boldsymbol{m_9}, m_{10} \\
P(m_{15}) = \underline{w_{21}}, \boldsymbol{w_{15}} & P(w_{15}) = m_{18}, m_{17}, m_{16}, \boldsymbol{m_{15}}, m_{14}, m_{13}, m_{12}, m_{11}, \underline{m_{10}} \\
P(m_{16}) = \underline{w_{22}}, \boldsymbol{w_{16}} & P(w_{16}) = m_{18}, m_{17}, \boldsymbol{m_{16}}, m_{15}, m_{14}, m_{13}, m_{12}, \underline{m_{11}}, m_{10} \\
P(m_{17}) = \underline{w_{23}}, \boldsymbol{w_{17}} & P(w_{17}) = m_{18}, \boldsymbol{m_{17}}, m_{16}, m_{15}, m_{14}, m_{13}, \underline{m_{12}}, m_{11}, m_{10} \\
P(m_{18}) = \underline{w_{24}}, \boldsymbol{w_{18}} & P(w_{18}) = \boldsymbol{m_{18}}, m_{17}, m_{16}, m_{15}, m_{14}, \underline{m_{13}}, m_{12}, m_{11}, m_{10} \\
\vdots & \vdots
\end{array}
$$

Figure 2: Visualization of the market used in the proof of Theorem 4.4. Underlined agents indicate matches under μ; bolded agents indicate matches under μ_M. $\succ$ is replaced with a comma and $\emptyset$ at the end of each preference list is omitted for space.

Theorem 4.3 (Roth, [2])**.** *In a finite, one-to-one market, there is no individually rational matching μ such that $\mu \succ_M \mu_M$.*

However, we show that the conclusion of Theorem 4.3 does not hold in infinite markets.

Theorem 4.4. *In an infinite, one-to-one market, there may be an individually rational matching μ such that $\mu \succ_M \mu_M$.*

Proof. Consider the following infinite market I where each agent is indexed with a positive integer and T_i is the i^{th} triangular number:

$m_{T_i+k} : w_{T_{i+1}+k} \succ w_{T_i+k} \succ \emptyset$ for $0 \leq k \leq i-2$,

$m_{T_i+k} : w_{T_{i+1}} \succ w_{T_{i+1}+1} \succ \ldots \succ w_{T_{i+1}+k+1} \succ \emptyset$ for $i-1 \leq k \leq i$,

for $i = 1, w_{T_i+k} : \emptyset$ for $0 \leq k \leq i$,

for $i \geq 2,\ w_{T_i+k} : m_{T_{i-1}+2i-2} \succ m_{T_{i-1}+2i-1} \succ \ldots \succ m_{T_{i-1}} \succ \emptyset$ for $0 \leq k \leq i-2,$ and

$w_{T_i+k} : m_{T_i+k-i-1} \succ m_{T_i+k-i} \succ \emptyset$ for $i-1 \leq k \leq i$.

Note that because $T_{i+1} - T_i = i+1$ by the definition of triangular numbers, the preferences of each agent are well defined. The market can be visualized as in Figure 2.

The man-optimal stable matching μ_M can be stated as follows:

$$\mu_M(m_{T_i+k}) = w_{T_i+k} \text{ for } 0 \leq k \leq i-2,$$

$$\mu_M(m_{T_i+k}) = w_{T_{i+1}+k+1} \succ \emptyset \text{ for } i-1 \leq k \leq i.$$

To show that μ_M is man-optimal, we first verify that it is stable. By checking the preferences, we see that μ_M is individually rational. In addition, μ_M is block-free, as each woman gets matched to her top choice man among those that list her in their preferences. Now, we show that μ_M is man-optimal. Assume the contrary and that μ'_M $(\neq \mu_M)$ instead is man-optimal. We will show that μ'_M cannot exist through a series of three steps:

1. $\mu'_M(m_{T_i}) = w_{T_i}$;

2. $\mu'_M(m_{T_i+k}) = w_{T_i+k}$ for $1 \leq k \leq i-2$;

3. $\mu'_M(m_{T_i+k}) = w_{T_{i+1}+k+1}$ for $i-1 \leq k \leq i$.

If all these steps are true, then $\mu_M = \mu'_M$, a contraction to our assumption. The proofs for each of these claims are attached in Appendix A. Yet, while μ_M is man-optimal, it is not Pareto-optimal. Indeed, the following matching μ

$$\mu(m_{T_i+k}) = w_{T_{i+1}+k} \text{ for } 0 \leq k \leq i-2,$$

$$\mu(m_{T_i+k}) = w_{T_{i+1}+k} \text{ for } i-1 \leq k \leq i.$$

is individually rational, and each man prefers his match under μ than in μ_M. $\square$

Demange et al. [15] showed linkages between stable matchings and solutions concepts in cooperative game theory. Our counterexample to the weak Pareto optimality principle implies additionally that the strong stability property from the finite market does not carry over to the infinite market. See Appendix E for details.

5 Conclusion

In Section 3, we showed that the man-optimal stable matching mechanism in infinite markets, like in finite markets, is group strategyproof (Theorem 3.3), satisfies natural entry comparative statics (Theorem 3.5), respects unambiguous improvement (Theorem 3.7), and respects preference extensions (Theorem 3.10). In addition, we showed that infinite markets carry over the Conway [18] and Hatfield-Kominers [21] lattice structures from finite markets (Appendix B, C). In Section 4, we proved the failure of weak Pareto optimality (Theorem 4.4) and strong stability property (Appendix E) in infinite markets.

Further potential directions of research involve verifying whether Blair's result holds [23] in infinite markets. Blair proved that in finite markets, every finite distributive lattice is equal to the lattice of stable matchings of some marriage market. Zanardo [12] conjectured that such a result does not hold true as the number of stable matchings in infinite markets should be either finite or uncountably infinite. However, beyond this conjecture, Blair's result in infinite markets remains an open problem. Another potential direction of research is to construct a systemic method that can identify all of the stable matchings in infinite markets, potentially mimicking the mechanism in finite markets illustrated by Roth and Sotomayor [2]. In addition, while most of the discussion in Section 3 of this paper was based on the assumption that each agent has a countably infinite list of acceptable partners, a potential future direction is to remove this constraint.

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Appendix

A Proofs omitted from the main text

Proof of Lemma 3.1

First, μ in I' is individually rational because for μ to have been a stable matching in the infinite economy, it must have matched each agent to partners they find acceptable. In addition, μ in I' must not contain blocking pairs as potential blocking pairs are only reduced from the ones in I. In other words, if (m, w) form a blocking pair in I' under μ, then they must form a blocking pair in I as well.

Proof of Theorem 3.3

Say that $\acute{I} = \{m_1, m_2, \ldots, m_k, w_1, w_2, \ldots, w_\ell\}$ and that a stable matching under manipulation is μ. We wish to show that if agents in $\acute{I}$ report their true preferences, then there is some stable matching in which at least one agent of $\acute{I}$ gets matched to an agent at least as truly preferable as their partners under μ. If under μ, an agent $i \in \acute{I}$ is not matched to a partner, then our claim holds true. Therefore, as we proceed, we say that each agent in $\acute{I}$ is matched to a partner in μ.

We first note that under the manipulation, if each agent in $\acute{i} \in \acute{I}$ even further manipulates their preferences to $\mu(\acute{i}) \succ \emptyset$, then μ still remain a stable matching. The matching is still individually rational as the pairings do not change. In addition, the matching remains block free as the potential blocks are only reduced from the ones before further manipulation.

Next, we generate the set of formulae Φ' that contains the following formula in addition to our usual set Φ. That is:

$$\begin{aligned}&(\vee_{w \succeq_{m_1} \mu(m_1)} \mathsf{matched}[m_1, w]) \vee \ldots \vee (\vee_{w \succeq_{m_k} \mu(m_k)} \mathsf{matched}[m_k, w]) \\ &\vee (\vee_{m \succeq_{w_1} \mu(w_1)} \mathsf{matched}[m, w_1]) \vee \ldots \vee (\vee_{m \succeq_{w_\ell} \mu(w_\ell)} \mathsf{matched}[m, w_\ell]) \in \Phi',\end{aligned} \tag{2}$$

which ensures that at least one agent in $\acute{I}$ is matched to an agent at least as truly preferable their partners under μ. Note that Eq. (2) is finite because there are only a finite number of agents that each agent in the finite set $\acute{I}$ prefers over their matches under μ.

Now, take a finite number of formulae from Φ', and let $M', W', \mu(M'), \mu(W')$, $\acute{I}$, and $\mu(\acute{I})$ be a finite subset I' of I where M' and W' are the set of men and women mentioned in our finite set of formulae. We will show that the man-optimal stable matching μ'_M under true preferences in I' satisfies our finite subset of formulae from Φ. Notice that as long as we have a stable matching, we satisfy all the formulae except for potentially the new addition, Eq. (2). To satisfy Eq. (2) , we should show that at least one agent from $\acute{I}$ is matched with a partner at least as preferable as their partners under μ.

Notice that the agents in I' can manipulate their preferences such that μ is a stable matching in this economy. (In particular, each agent $\acute{i} \in \acute{I}$ can reduce their preferences to $\mu(\acute{i}) \succ \emptyset$ and by Lemma 3.1, μ will be a stable matching of I'.) As I' is a finite economy, by Theorem 3.2, in μ'_M, there must be least one agent from $\acute{I}$ that gets matched to a partner as preferable as their partners under μ. Therefore, μ'_M satisfies all the equations in the finite set of our formulae. As we have shown that there is a model that satisfies any finite set of formulae in Φ', we know that there is a model satisfying every formula in Φ' by logical compactness. Thus, we have shown that when each agent $i \in \acute{I}$ reports their true preferences, at least one agent can be matched to a partner at least as preferable as their partners post manipulation as we had sought to prove.

Proof of Theorem 3.7

Let the women's preferences after man $\overline{m}$ unambiguously improves be their *updated preferences*. Say that μ_M is the man-optimal stable matching prior to the update. We want to show that under updated preferences, $\overline{m}$ gets man-optimally matched to a woman as preferable as $\overline{w}$. If $\mu_M(\overline{m}) = \emptyset$, our claim holds. Therefore, we assume that $\mu_M(\overline{m}) = \overline{w}$.

We once again construct a new set of formulae Φ', which contains our usual formulae Φ and the following:

$$\vee_{w \succeq_{\overline{m}} \overline{w}} \mathsf{matched}[\overline{m}, w] \in \Phi', \tag{3}$$

which ensures that $\overline{m}$ is matched to a woman at least as preferable as $\overline{w}$. The formula is finite as man $\overline{m}$ can only prefer finitely many women over $\overline{w}$.

Let us take a finite set of formula from Φ' and let our resulting economy I' be M', W', $\mu_M(M')$, $\mu_M(W')$, $\overline{m}$, and $\overline{w}$, where M' and W' are the finite set of men and women mentioned in our finite set of formulae. We will show that the man-optimal stable matching μ'_M in I' post-update satisfies our finite subset of formula. First note that since such a matching is stable, we only need to show that it satisfies Eq. (3). From Lemma 3.1, we have that μ_M is a stable matching in I' prior to the update. From here, we deduce that the man-optimal stable matching in I' prior to the update matches $\overline{m}$ to a woman at least as preferable as $\mu_M(\overline{m}) = \overline{w}$. Now, note that the unambiguous improvement of $\overline{m}$ in the infinite economy implies that $\overline{m}$ unambiguously improves in I' as well because I' is a subset of I. And by Theorem 3.6, $\overline{m}$ gets matched to a woman at least as preferable as $\overline{w}$ in μ'_M. Thus, μ'_M satisfies our finite set of formulae. Therefore, we have shown that there is a model that satisfies any finite subset of formula in Φ'. Thus, by logical compactness, there exists a model that satisfies all the formulae in Φ', which we had sought to prove.

Proof of Theorem 3.10

We will first show that $\mu_M \succeq_M \tilde{\mu}_M$. We will say that M *updates* their preferences when they modify their preferences from P to $\tilde{P}$. Suppose that $\widehat{M}$ is the set of men matched to some woman under $\tilde{P}$. To prove our claim, we need to show that under P, every man $\widehat{m} \in \widehat{M}$ gets matched to a woman at least as preferable as $\tilde{\mu}_M(\widehat{m})$. To force this assertion on top of our usual set of formulae Φ, which alone, guarantees a stable matching, we add the following formulae to Φ for all men $\widehat{m} \in \widehat{M}$ and construct a new set of formulae Φ':

$$\vee_{w \succeq_{\widehat{m}} \tilde{\mu}_M(\widehat{m})} \mathsf{matched}[\widehat{m}, w] \in \Phi', \tag{4}$$

which ensures that each $\widehat{m} \in \widehat{M}$ is matched to a woman at least as preferable as $\tilde{\mu}_M(\widehat{m})$. Each of the formulae added is finite in length, because each man $\widehat{m}$ can only prefer finitely many woman ahead of $\tilde{\mu}_M(\widehat{m})$.

Let us take a finite set of formulae from Φ', and let our finite economy I' consist of $M', W', \tilde{\mu}_M(M')$, and $\tilde{\mu}_M(W')$. We will show that the man-optimal stable matching after update μ'_M in I' will satisfy our subset of formulae from Φ'. Note that because such a matching is already stable by definition, the only formulae that we should satisfy in addition are Eq. (4). In other words, we want to ensure that in the man-optimal stable matching of I', every man $m \in M' \cup \widehat{M}$ is matched to a man at least as weakly preferable as their partners under $\tilde{\mu}_M$. First, by Lemma 3.1, we know that $\tilde{\mu}_M$ is a stable matching in I'. By definition of man-optimal, each man $m' \in M'$ will be matched to a man at least weakly more preferable than $\tilde{\mu}_M(m')$ under the man-optimal stable matching pre-update. Now, say that each man updates his preferences. Because I' is a finite economy, by Theorem 3.9, we know that the man-optimal stable matching under updated preferences should match each man with a partner at least weakly more preferable than their partners under $\mu'_{M'}$. In particular, each man $m \in M' \cap \widehat{M}$ is matched to a woman at least as preferable as $\tilde{\mu}_M(m)$. Therefore, as we found a stable matching that satisfies any finite subset of formulae extracted from Φ', by logical compactness, we know that there exists a model that satisfies all of Φ'.

We will now show that $\tilde{\mu}_W \succeq_W \mu_W$. We will say that M *outdates* their preferences when they modify their preferences from $\tilde{P}$ back to P. Suppose that $\check{W}$ is the set of women matched to some man under P. To prove our claim, we need to show that under $\tilde{P}$, every woman $\check{w} \in \check{W}$ gets matched to a man at least as preferable as $\mu_W(\widehat{w})$. To force this assertion on top of our usual set of formulae Φ, which alone, guarantees a stable matching, we add the following formulae for all women $\check{w} \in \check{W}$ to Φ and construct a new set of formulae Φ':

$$\vee_{m \succeq_{\check{w}} \mu_W(\check{w})} \mathsf{matched}[m, \check{w}] \in \Phi', \tag{5}$$

which ensures that each $\check{w} \in \check{W}$ is matched to a man at least as preferable as $\mu_W(\check{w})$. Each of the formulae is finite in length, because $\check{w}$ can only prefer finitely many men over $\mu_W(\check{w})$.

Let us take a finite set of formulae from Φ', and let our finite economy I' consist of $M', W', \mu_W(M')$, and $\mu_W(W')$ where M' and W' are the finite set of men and women mentioned in the finite set of formulae. We will show that the woman-optimal stable matching after update in this finite economy μ'_W will satisfy our subset of formulae from Φ'. Note that because such a matching is already stable, the only formulae that we should satisfy in addition is Eq. (5). In other words, we want to ensure that μ'_W, every woman $w \in W' \cup \check{W}$ is matched to a man at least as weakly preferable as their partners under μ_W. First, by Lemma 3.1, we know that μ_W is a stable matching in I'. By definition, each woman $w' \in W'$ will be matched to a man at least as weakly more preferable as her match in the woman-optimal stable matching in I' post-update. Now, say that each woman outdates her preferences. By Theorem 3.9, we know that the woman-optimal stable matching under outdated preferences should match each woman with a partner at least as weakly more preferable than the partners under $\mu'_{W'}$. In particular, each woman $w \in W' \cap \check{W}$ is matched to a man at least as preferable as $\mu_W(w)$. Therefore, as we found a stable matching that satisfies any finite subset of formulae extracted from Φ', by logical compactness, we know that there exists a model that satisfies all of Φ'.

Counterexample to Theorem 4.2 with Uncountably Infinite Agents

Consider the following market in which every man and woman are indexed with a real number in the interval $[0, 1)$:

$m_{(\frac{1}{2})^k} : w_{(\frac{1}{2})^k} \succ w_{(\frac{1}{2})^{k-1}} \succ \emptyset$ for $k \in \mathbb{Z}^+$, $m_r : w_r \succ \emptyset$ for $r \notin \{(\frac{1}{2})^k \mid k \in \mathbb{Z}^+\}$ and

$w_{(\frac{1}{2})^k} : m_{(\frac{1}{2})^{k+1}} \succ m_{(\frac{1}{2})^k} \succ \emptyset$ for $k \in \mathbb{Z}^+, w_r : m_r \succ \emptyset$ for $r \notin \{(\frac{1}{2})^k \mid k \in \mathbb{Z}^+\}$.

Consider μ_M and μ_W in this economy. Similar to μ_M and μ_W in Example 4.1, μ_M (respectively, μ_W) matches every man (respectively, woman) to their top choices. These matchings are individually rational and contain no blocking pairs as every man (respectively, woman) prefers no one above their partners. Notice that in μ_M, every man is matched to a woman, while in μ_W, $m_{\frac{1}{2}}$ does not have a partner.

Omitted Steps from the Proof of Theorem 4.4

Claim A.1. $\mu'_M(m_{T_i}) = w_{T_i}$

Assume the contrary and that $\mu'_M(m_{T_i}) \neq w_{T_i}$. Because we already know that μ_M is a stable matching in I and that m_{T_i} is matched to a woman in μ_M, it cannot be the case that $\mu'_M(m_{T_i}) = \emptyset$ by our assumption that μ'_M is man-optimal. Therefore, $\mu'_M(m_{T_i}) = w_{T_{i+1}}$. However, if so, $w_{T_{i+1}}$ and m_{T_i+i-1} create a blocking pair as $w_{T_{i+1}}$ prefers m_{T_i+i-1} over m_{T_i}, and m_{T_i+i-1} ranks $w_{T_{i+1}}$ as his most preferable partner. Thus, our assumption that $\mu'_M(m_{T_i}) \neq w_{T_i}$ cannot be true.

Claim A.2. $\mu'_M(m_{T_i} + k) = w_{T_i+k}$ *for* $1 \leq k \leq i - 2$

We proceed via strong induction on k while allowing i to vary across positive integers such that $k \leq i$. We let the base case be $k = 0$, which we know holds true for all i through Claim A.1. For our strong inductive step, we say that for all $\ell \leq k$, $\mu'_M(m_{T_i+\ell}) = w_{T_i+\ell}$ for all $k \leq i$. Our goal is to show that $\mu'_M(m_{T_i+k+1}) = w_{T_i+k+1}$. Assume the contrary. As we know that μ_M is stable and that $\mu_M(m_{T_i+k+1}) = w_{T_i+k+1}$, for μ'_M to be man-optimal, it must be that $\mu'_M(m_{T_i+k+1}) = w_{T_{i+1}+k+1}$. Now, our strong induction hypothesis implies that $\mu_M(m_{T_{i+1}+\ell}) = w_{T_{i+1}+\ell}$ for $0 \leq \ell \leq k$. Therefore, $w_{T_{i+1}+\ell}$ for $0 \leq \ell \leq k$ is not available for m_{T_i+i-1}, leaving $w_{T_{i+1}+k+1}$ as the top available candidate in m_{T_i+i-1}'s preferences. Also, under $w_{T_{i+1}+k+1}$'s preferences, m_{T_i+i-1} is ranked higher than m_{T_i+k}. Therefore, m_{T_i+i-1} and $w_{T_{i+1}+k+1}$ form a blocking pair, meaning that our assumption, $\mu'_M(m_{T_i+k}) \neq w_{T_i+k}$, cannot be true. Thus, $\mu'_M(m_{T_i+k}) = w_{T_i+k}$ as we had sought.

Claim A.3. $\mu_M(m_{T_i+k}) = w_{T_{i+1}+i+k-2}$ *for* $i-1 \leq k \leq i$

As m_{T_i+k} for $i-1 \leq k \leq i$ are matched to some woman under the stable matching μ_M, under the supposedly man optimal μ'_M, these men are matched to some woman as well. But because we know that Claim A.1 and Claim A.2 hold true, we know that $w_{T_{i+1}}, w_{T_{i+1}+1} \ldots w_{T_{i+1}+i-1}$ are unavailable to m_{T_i+k} for $i-1 \leq k \leq i$ as they are already taken. Therefore, the only available woman for m_{T_i+i-1} is $w_{T_{i+1}+i}$ and the only available woman for m_{T_i+i} is $w_{T_{i+1}+i+1}$. Therefore, $\mu'_M(m_{T_i+k}) = w_{T_{i+1}+k-1}$ for $i-1 \leq k \leq i$ as we had sought to prove.

B Conway Lattice

To interpret Conway's Lattice in the context of fixed points, Adachi [19] defined a *prematching* $v = (v_M, v_W)$ where $v_M(m)$ maps each man m to himself or to a woman and $v_W(w)$ maps each woman w to herself or to a man. In Adachi's words, a matching μ is said to *define* a prematching v if for all $m \in M$ and $w \in W$, $v_M(m) = \mu(m)$ and $v_W(w) = \mu(w)$.

Adachi characterized the stable matchings μ to be matchings defined by v that satisfy the following

$$v_M(m) = \max_{\succ m}\{w \in W \mid m \succeq_w v_W(w) \cup \{m\}\} \tag{6}$$

$$v_W(w) = \max_{\succ w}\{m \in M \mid m \succeq_m v_M(m) \cup \{w\}\} \tag{7}$$

and showed that the solutions of Eq. (6) and Eq. (7), i.e., the set of stable matchings, form a complete lattice. Here, we note that Adachi's setup carries over to infinite markets.

Proof of Theorem 3.12

The proof follows that of Adachi's [19] directly. The two instances in which Adachi makes use of the finiteness of the market in his proof is in (1) defining Eq. (6) and Eq. (7) and (2) applying Tarski's Fixed Point Theorem [20]. In both instances, the preference list of each agent being countable and having a well-defined most preferable agent allows for the equations to be well defined and for the lattice to be complete. Such results are grounded in the following idea. Suppose that we have a countably infinite set with well-ordered elements $e_1 < e_2 < \ldots$ and a clear minimum value e_i. If we choose a single element from the set (potentially uncountably) infinitely many times, the minimum element among the chosen is well defined as choosing elements from the set is equivalent to taking a subset of the original set after removing duplications. As the original set is countably infinite, well-ordered, and has a well-defined minimum element, its subset does have a well-defined minimum element as well.

C Decomposition Lemma

Statement in Finite Markets

The statement of the decomposition lemma in finite markets was first proposed by Gale and Sotomayor [16].

Lemma C.1 (Gale and Sotomayor [16]). *In a finite, one-to-one market, let μ and μ' be two stable matchings. Let $M(\mu)$ denote the set of men that prefer their partners under μ than μ'. Similarly, define $M(\mu')$, $W(\mu)$, and $W(\mu')$. Then, μ and μ' both map $M(\mu')$ onto $W(\mu)$ and similarly, $M(\mu)$ to $W(\mu')$.*

Proof of Lemma 3.13

We first prove the following claim that will help us prove the rest of the result.

Claim C.1. *$\mu(M(\mu)) \in W(\mu')$. Similarly, we have that $\mu'(M(\mu')) \in W(\mu)$, $\mu(W(\mu)) \in M(\mu')$, and $\mu'(W(\mu')) \in M(\mu)$.*

We show that $\mu(M(\mu)) \in W(\mu')$. The others follow in the same way. Assume otherwise, which is that $\mu(M(\mu)) \in W(\mu)$. Suppose that m is in $M(\mu)$. Because $m \in M(\mu)$, $\mu(m) \succ_m \mu'(m)$. Similarly, because we assumed that $\mu(m) \in W(\mu)$, $m \succ_{\mu(m)} \mu'(\mu(m))$. However, this yields a contradiction as m and $\mu(m)$ form a blocking pair under μ'. Therefore, it must be the case that $\mu(M(\mu)) \in W(\mu')$.

From Claim C.1 alone, we know that μ maps $M(\mu)$ to $W(\mu')$. We will now show that μ maps $W(\mu')$ to $M(\mu) \cup \{\emptyset\}$. Let $\check{\lambda} = \mu \vee \mu'$. By Theorem 3.12, we know that $\check{\lambda}$ is a stable matching. In particular, this implies that $\check{\lambda}(m) = w$ if and only if $\check{\lambda}(w) = m$. By the definition of $W(\mu')$, for every $w \in W(\mu')$, $\check{\lambda}(w) = \mu(w)$. Assume that $\mu(w) \in M(\mu)$. Then, $\check{\lambda}(\mu(w)) = \mu(\mu(w)) \in W(\mu)$ by Claim C.1. Therefore, $\check{\lambda}(\mu(w)) \neq w$, a contradiction as $\check{\lambda}(w) = \mu(w)$. Therefore, $\mu(w) \notin M(\mu)$, and μ' maps $M(\mu) \cup \{\emptyset\}$ to $W(\mu')$. Likewise, considering the pair $M(\mu')$ and $W(\mu)$ (instead of $M(\mu)$ and $W(\mu')$) and $\hat{\lambda} = \mu \wedge \mu'$ (instead of $\check{\lambda} = \mu \vee \mu'$) finishes all directions of the proof.

D Hatfield-Kominers Lattice

Hatfield and Kominers [21] discussed lattice in the context of finite many-to-one matchings, expanding upon the model proposed by Hatfield and Milgrom [24]. Here, we restrict their model to the setting of one-to-one matching and lift their results to infinite markets.

To define the Hatfield-Kominers lattice in the context of a one-to-one marriage model, we let X denote the set of every pairing of man and woman (m, w) for $m \in M$ and $w \in W$. In addition, for any subset of pairs $Y \subseteq X$, we let Y_m denote set of pairs $y \in Y$ that matches man m with some woman. Y_w can be defined in a similar fashion. For any set of pairs $Y \subseteq X$, we let $C_M(Y) = \bigcup_{m \in M} \max_{\succ_m}\{y \in Y_m\}$. Putting into words, $C_M(Y)$ is the set of most preferable pairs for each man among those in Y. C_W is defined in the same fashion.

The Hatfield-Kominers operator Φ in marriage matching can be described as follows:

$$\Phi(X^M, X^W) = (\Phi_M(X^W), \Phi_W(X^M)),$$

$$\text{where } \Phi_M(X^W) = \{x \in X : x \in C_W(X^W \cup \{x\})\}$$

$$\text{and } \Phi_M(X^W) = \{x \in X : x \in C_M(X^M \cup \{x\})\}.$$

To put it in words, at each iteration of Φ, the men and women modify X_M and X_W to the set of each of their most preferable pairs based on X_W and X_M from the previous iteration. Hatfield and Kominers showed that at the set of fixed points of Φ, i.e., points in which X_M and X_W both remain fixed after another iteration of Φ on them, $X_M \cap X_W$ yields stable matchings. In addition, they showed that Φ forms a lattice when

$$\Phi(X^M, X^W) \wedge \Phi(X'^M, X'^W) = \Phi(X^M \cup X'^M, X^W \cap X'^W) \text{ and}$$

$$\Phi(X^M, X^W) \vee \Phi(X'^M, X'^W) = \Phi(X^M \cap X'^M, X^W \cup X'^W).$$

Here, we note that the same result holds in infinite markets.

Theorem D.1. *In a (potentially infinite) market, the fixed points under Φ correspond to stable matchings and form a lattice.*

Proof. The proof follows the proof given by Hatfield and Kominers directly as (1) Φ remains well defined and (2) the lattice remains complete as the union and joint of subsets of an infinite set are subsets of the original set. □

E Strong Stability Property

Another way to view matching is through the lens of cooperative game theory. In cooperative game theory, the *core* of the game is the set of undominated outcomes. In the context of matching, domination can be defined as follows: a matching μ' dominates another matching μ defined under the same M, W, and P if there exists a finite set of agents $I' \in I$ such that for all men $m \in I'$ and all women $m \in I'$, $\mu'(m) \in I'$, $\mu'(w) \in I'$, $\mu'(m) \succ_m \mu(m)$, and $\mu'(w) \succ_w \mu(w)$. It is a classic result by Roth and Sotomayor [2] that the core of the one-to-one marriage model is the set of stable matchings in finite markets. Such a result comes at a surprise as the same result does not hold true for many-to-one or many-to-many matchings. However, it is the case in infinite one-to-one markets that the set of stable matchings remain as the core of the marriage market.

Theorem E.1. *In a (potentially infinite), one-to-one market, the core of the matchings in the marriage model is the set of stable matchings.*

Proof. The proof follows the proof provided by Roth and Sotomayor in finite markets [2] directly. □

A natural question that arises from such a result is whether there always exists a stable matching that dominates μ for an unstable individually rational matching μ. It turns out that in finite markets, this is nearly the case.[5] Demange, Gale, and Sotomayor [15] proved the following statement, dubbed as the strong stability property, that formalizes this notion.

[5]Nearly as the preferences in Theorem E.2 are not strict.

Theorem E.2 (Demange, Gale, and Sotomayor [15]). *In a finite, one-to-one market, if μ is an unstable individually rational matching, then there exists a blocking pair (m, w) and a stable matching $\tilde{\mu}$ such that $\tilde{\mu}(m) \succeq \mu(m)$ and $\tilde{\mu}(w) \succeq \mu(w)$.*

However, despite that the set of stable matchings remain as the core of the marriage model in infinite markets, the strong stability property does not hold in infinite markets.

Theorem E.3. *In an infinite, one-to-one market, if μ is an unstable individually rational matching, then there need not exist a blocking pair (m, w) and a stable matching $\tilde{\mu}$ such that $\tilde{\mu}(m) \succeq \mu(m)$ and $\tilde{\mu}(w) \succeq \mu(w)$.*

Proof. Refer to the example in our proof of Theorem 4.4. We will show that μ_M is the only stable matching in this economy. We first examine that μ_M coincides with the woman-optimal stable matching as in this economy as in μ_M, every woman is matched to their top choice men among those that list her in his preferences. Therefore, by Theorem 3.12, which states that the set of stable matchings form a lattice and that the man-optimal and woman-optimal stable matchings are the max and min elements of this lattice, we conclude that μ_M is the only stable matching in this economy. Now, let $\tilde{\mu}$ in the statement of Theorem E.3 be μ_M. As no man m satisfies $\tilde{\mu}(m) = \mu_M(m) \succ \mu(m)$, Theorem E.2 does not hold true in infinite markets. □